

MTH250 - CALCULUS

Lecture No. 21

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These lecture notes are prepared to serve as handouts for the video lectures on the course *MTH-250 Calculus* of Virtual Campus, COMSATS Institute of Information Technology, Islamabad, Pakistan. The notes were derived from an extensive collection of notes and books. Most of the theoretical details as well as worked problems are from *Calculus*, by *H. Anton, I Bivens and S. Davis*, John Wiley & Sons, 10th edition, 2012.

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LECTURE 21

Improper integrals & Introduction to vectors

21.1 Improper integrals

In defining a definite integral $\int_a^b f(x) dx$ we dealt with a function f defined on a finite interval $[a, b]$ and we assumed that f does not have an *infinite discontinuity*.

Definition 21.1 (Improper integrals: Informal).

An integral is called improper integral if either the interval is infinite, or the function f has an infinite discontinuity in $[a, b]$.

One of the most important applications of this idea is in probability distributions and integral transformations of functions such as Fourier and Laplace transforms.

Example 21.1. Find the area of the region S bounded under the curve $y = \frac{1}{x^2}$, above the x -axis and by the line $x = 1$ to the right; see Figure 21.1.

Solution. The area of region S that lies to the left of the line $x = t$ is:

$$A(t) = \int_1^t \frac{1}{x^2} dx = -\frac{1}{x} \Big|_1^t = 1 - \frac{1}{t}$$

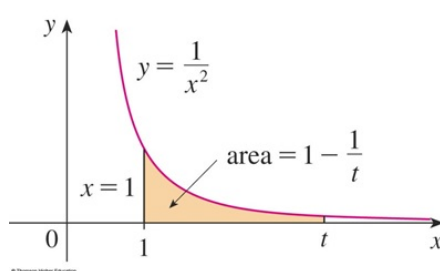


Figure 21.1: Region S bounded under the curve $y = \frac{1}{x^2}$ above the x -axis and by the line $x = 1$ to the right.

Notice that $A(t) < 1$ no matter how large t is chosen. We also observe that

$$\lim_{t \rightarrow \infty} A(t) = \lim_{t \rightarrow \infty} \left(1 - \frac{1}{t}\right) = 1.$$

The area of the shaded region approaches 1 as $t \rightarrow \infty$. So, we say that the area of the infinite region S is equal to 1 and we write:

$$\int_1^{\infty} \frac{1}{x^2} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x^2} dx = 1.$$

□

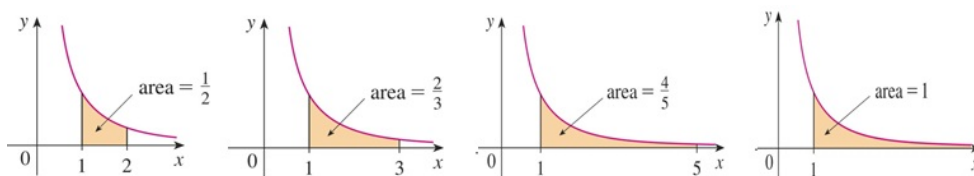


Figure 21.2:

21.1.1 Improper integral of type I

Definition 21.2 (Improper integral of type I).

If $\int_a^t f(x) dx$ exists for every number $t \geq a$, then

$$\int_a^{\infty} f(x) dx = \lim_{t \rightarrow \infty} \int_a^t f(x) dx$$

provided this limit exists (as a finite number).

If $\int_t^b f(x)dx$ exists for every number $t \geq b$, then

$$\int_{-\infty}^b f(x)dx = \lim_{t \rightarrow -\infty} \int_t^b f(x)dx$$

provided this limit exists (as a finite number).

An improper integral is called *convergent* if the corresponding limit exists. Otherwise it is *divergent*.

If both $\int_a^\infty f(x)dx$ and $\int_{-\infty}^a f(x)dx$ are convergent, then we define:

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx$$

where a can be any real number.

Example 21.2. Determine whether the integral $\int_1^\infty \frac{1}{x} dx$ is convergent or divergent.

Solution. By definition

$$\int_1^\infty \frac{1}{x} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x} dx = \lim_{t \rightarrow \infty} \ln|x| \Big|_1^t = \lim_{t \rightarrow \infty} (\ln t - \ln 1) = \lim_{t \rightarrow \infty} \ln t = \infty.$$

The limit does not exist as a finite number. So, the integral is divergent. $\square$

Example 21.3. Evaluate $\int_{-\infty}^0 xe^x dx$.

Solution. By definition

$$\int_{-\infty}^0 xe^x dx = \lim_{t \rightarrow -\infty} \int_t^0 xe^x dx.$$

We integrate by parts with $u = x$ and $dv = e^x dx$. This implies that $du = dx$ and $v = e^x$. Therefore,

$$\int_t^0 xe^x dx = xe^x \Big|_t^0 - \int_t^0 e^x dx = -te^t - 1 + e^t.$$

We know that $te^t \rightarrow 0$ as $t \rightarrow -\infty$ therefore by l'Hospital rule

$$\lim_{t \rightarrow -\infty} te^t = \lim_{t \rightarrow -\infty} \frac{t}{e^{-t}} = \lim_{t \rightarrow -\infty} (-e^t) = 0$$

Therefore,

$$\int_{-\infty}^0 xe^x dx = \lim_{t \rightarrow -\infty} (-te^t - 1 + e^t) = -0 - 1 + 0 = -1.$$

$\square$

Example 21.4. Evaluate $\int_{-\infty}^\infty \frac{1}{1+x^2} dx$.

Solution. By definition

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \int_{-\infty}^0 \frac{1}{1+x^2} dx + \int_0^{\infty} \frac{1}{1+x^2} dx.$$

We must evaluate both integrals separately, *i.e.*

$$\begin{aligned} \int_0^{\infty} \frac{1}{1+x^2} dx &= \lim_{t \rightarrow +\infty} \int_0^t \frac{dx}{1+x^2} = \lim_{t \rightarrow +\infty} \tan^{-1} x \Big|_0^t \\ &= \lim_{t \rightarrow +\infty} (\tan^{-1} t - \tan^{-1} 0) = \lim_{t \rightarrow +\infty} \tan^{-1} t = \frac{\pi}{2}. \end{aligned}$$

and

$$\begin{aligned} \int_{-\infty}^0 \frac{1}{1+x^2} dx &= \lim_{t \rightarrow -\infty} \int_t^0 \frac{dx}{1+x^2} = \lim_{t \rightarrow -\infty} \tan^{-1} x \Big|_t^0 \\ &= \lim_{t \rightarrow -\infty} (\tan^{-1} 0 - \tan^{-1} t) = \lim_{t \rightarrow -\infty} (0 - \tan^{-1} t) = -\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}. \end{aligned}$$

Since both these integrals are convergent, the given integral is convergent and

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

As $\frac{1}{1+x^2} > 0$, the given improper integral can be interpreted as the area of the infinite region that lies under the curve $y = \frac{1}{1+x^2}$ and above the x -axis. $\square$

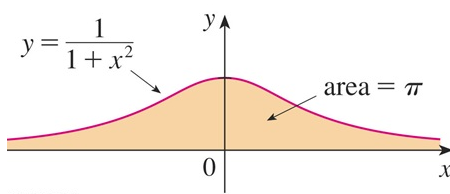


Figure 21.3:

21.1.2 Improper integral of type II

Let S be the unbounded region under the graph of f and above the x -axis between a and b . The area of the part of S between a and t is:

$$A(t) = \int_a^t f(x) dx$$

If it happens that $A(t)$ approaches a definite number A as $t \rightarrow b^-$, then we say that the area of the region S is A and we write:

$$\int_a^b f(x) dx = \lim_{t \rightarrow b^-} \int_a^t f(x) dx$$

provided this limit exists (as a finite number).

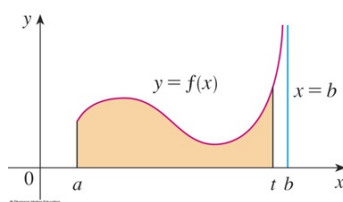


Figure 21.4: Improper integrals of type II

Definition 21.3 (Improper integral of type II). *If f is continuous on $[a, b)$ and is discontinuous at b , then*

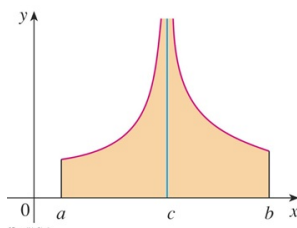
$$\int_a^b f(x) dx = \lim_{t \rightarrow b^-} \int_a^t f(x) dx$$

if this limit exists (as a finite number).

If f is continuous on $(a, b]$ and is discontinuous at a , then

$$\int_a^b f(x) dx = \lim_{t \rightarrow a^+} \int_t^b f(x) dx$$

provided this limit exists (as a finite number).

Figure 21.5: Discontinuity at $c \in (a, b)$

If f is discontinuous at $c \in (a, b)$ and if both $\int_a^c f(x) dx$ and $\int_c^b f(x) dx$, are convergent then

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$$

Example 21.5. Evaluate $\int_2^5 \frac{1}{\sqrt{x-2}} dx$.

Solution. First, we note that the given integral is improper because has the vertical asymptote $x = 2$. By definition,

$$\int_2^5 \frac{dx}{\sqrt{x-2}} = \lim_{t \rightarrow 2^+} \int_t^5 \frac{dx}{\sqrt{x-2}} = \lim_{t \rightarrow 2^+} 2\sqrt{x-2} \Big|_t^5 = \lim_{t \rightarrow 2^+} 2(\sqrt{3} - \sqrt{t-2}) = 2\sqrt{3}.$$

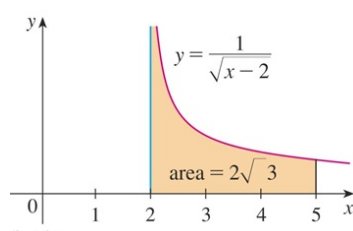


Figure 21.6: An improper integral of type II.

Thus, the given improper integral is convergent.

□

Example 21.6. Determine whether converges or diverges $\int_0^{\pi/2} \sec x dx$.

Solution. Note that the given integral is improper because

$$\lim_{x \rightarrow (\pi/2)^-} \sec x = \infty.$$

By definition

$$\begin{aligned} \int_0^{\pi/2} \sec x dx &= \lim_{x \rightarrow (\pi/2)^-} \int_0^t \sec x dx = \lim_{x \rightarrow (\pi/2)^-} \ln |\sec x + \tan x| \Big|_0^t \\ &= \lim_{x \rightarrow (\pi/2)^-} [\ln(\sec t + \tan t) - \ln 1] = \infty. \end{aligned}$$

Thus, the given improper integral is divergent. This is because $\sec t \rightarrow \infty$ and $\tan t \rightarrow \infty$ as $t \rightarrow (\pi/2)^-$.

□

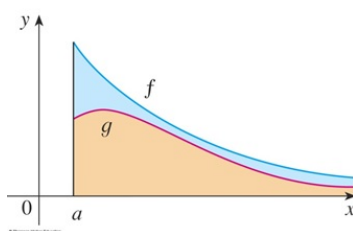
Sometimes, it is impossible to find the exact value of an improper integral and yet it is important to know whether it is convergent or divergent. In this case the following theorem is quite useful.

Theorem 21.4. Suppose f and g are continuous functions with

$$f(x) \geq g(x) \geq 0 \quad \text{for } x \geq a.$$

1. If $\int_a^\infty f(x) dx$ is convergent, then $\int_a^\infty g(x) dx$ is also convergent.
2. If $\int_a^\infty g(x) dx$ is divergent, then $\int_a^\infty f(x) dx$ is also divergent.

Example 21.7. Show that $\int_0^\infty e^{-x^2} dx$ is convergent.

Figure 21.7: $f(x) \geq g(x)$.

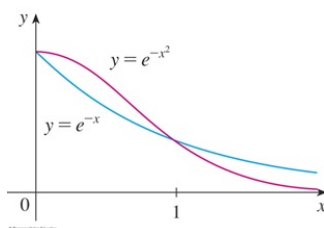
Example 21.8 (Solution.). We write

$$\int_0^{\infty} e^{-x^2} dx = \int_0^1 e^{-x^2} dx + \int_1^{\infty} e^{-x^2} dx.$$

We observe that the first integral on the right-hand side is just an ordinary definite integral. In second integral, we use the fact that, for $x \geq 1$, we have $x^2 \geq x$. So, $-x^2 \leq -x$ and, therefore, $e^{-x^2} \leq e^{-x}$. Moreover,

$$\int_1^{\infty} e^{-x} dx = \lim_{t \rightarrow \infty} \int_1^t e^{-x} dx = \lim_{t \rightarrow \infty} (e^{-t} - e^{-1}) = e^{-1}.$$

It follows that $\int_0^{\infty} e^{-x^2} dx$ is convergent since it is the sum of definite (with finite value) and convergent integrals.

Figure 21.8: Comparison of $y = e^{-x^2}$ and $y = e^{-x}$.

21.2 Introduction to vectors

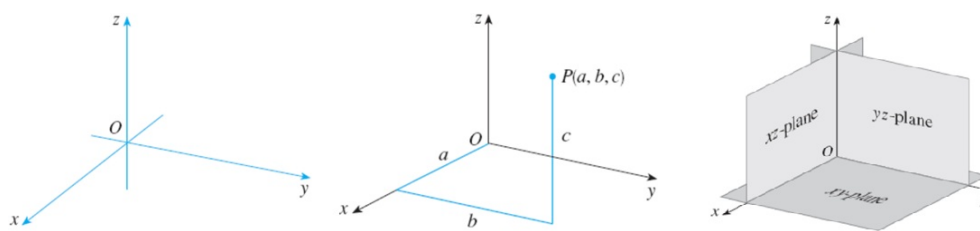


Figure 21.9: Left: xyz -coordinate system. Center: Locating a point $P(a, b, c)$ in xyz -coordinate system. Right: Coordinate planes.

Definition 21.5 (Distance formula).

The distance $|P_1 P_2|$ between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

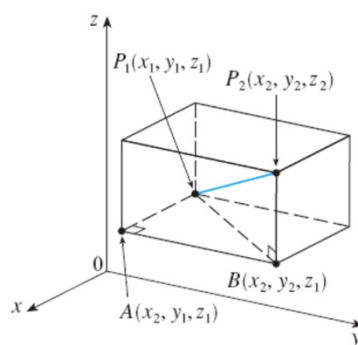


Figure 21.10: Distance between two points.

$$|P_1 P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Definition 21.6 (Vectors). The term *vector* is used by scientists to indicate a quantity that has both magnitude and direction, such as displacement or velocity or force. The displacement vector v , see Figure 21.11, has initial point A (the tail) and terminal point B (the tip) and we indicate this by writing $\vec{AB}$.

Notice that the vector $u = \vec{CD}$ has the same length and the same direction as v , even though it is in a different position. We say that u and v are equivalent (or equal) and we write $u = v$.

We have $\vec{AC} = \vec{AB} + \vec{BC}$.

Definition 21.7 (Sum & difference of vectors). If u and v are vectors positioned so the initial point of v is at the terminal point of u , then the sum $u + v$ is the vector from the initial point of u to the terminal point of v . By the difference $u - v$ of two vectors we mean $u - v = u + (-v)$.

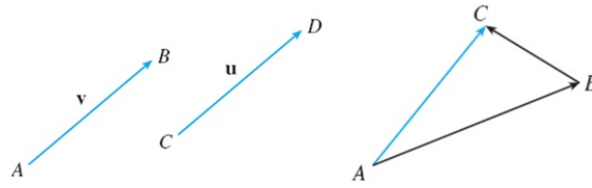


Figure 21.11: Distance between two points.

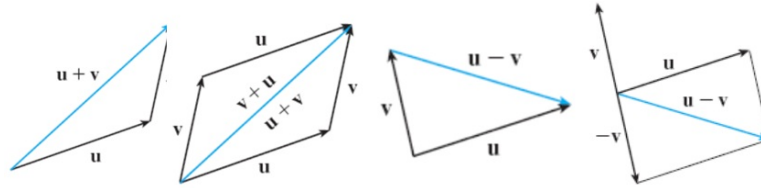


Figure 21.12: Sum, difference & shrinking of vectors.

Definition 21.8. If c is a scalar and v is a vector, then the scalar multiple cv is the vector whose length is $|c|$ times the length of v and whose direction is the same as v if $c > 0$ and is opposite to v . If $c < 0$. If $c = 0$ or $v = 0$, then $cv = 0$.

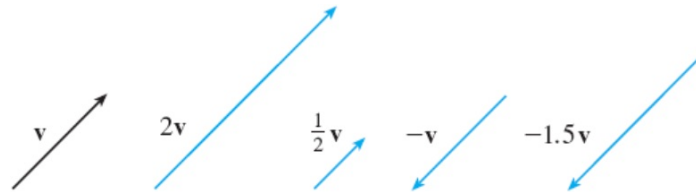


Figure 21.13: Shrinking & stretching of vectors.

Definition 21.9. Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ the vector from A to B with representation $\vec{AB}$ is given by

$$\vec{AB} = (x_2 - x_1, y_2 - y_1, z_2 - z_1) = \vec{OB} - \vec{OA}.$$

Definition 21.10 (Magnitude).

The length or the magnitude of the three-dimensional vector $\vec{a} = (a_1, a_2, a_3)$ is given by

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

Theorem 21.11 (Properties of vectors).

If $\vec{a}, \vec{b}$, and $\vec{c}$ are vectors in $\mathbb{R}^n$, $n > 1$ and k and s are scalars, then

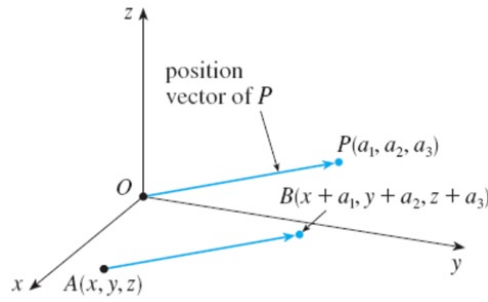


Figure 21.14: Position vectors.

1. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
2. $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$
3. $\vec{a} + \vec{0} = \vec{a}$
4. $\vec{a} + (-\vec{a}) = \vec{0}$
5. $k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$
6. $(k + s)\vec{a} = k\vec{a} + s\vec{a}$
7. $(ks)\vec{a} = k(s\vec{a}) = s(k\vec{a})$
8. $1\vec{a} = \vec{a}$.

Definition 21.12 (Unit vector). A vector v is called unit vector if it has unit length i.e. $|v| = 1$. The unit vectors along x, y, z -axes are respectively denoted by $\hat{i}, \hat{j}$ and $\hat{k}$, i.e.

$$\hat{i} = (1, 0, 0), \quad \hat{j} = (0, 1, 0), \quad \hat{k} = (0, 0, 1).$$

These vectors $\hat{i}, \hat{j}$, and $\hat{k}$ are called the standard basis vectors. Any vector $\vec{a} = (a_1, a_2, a_3)$ can be written as

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}.$$

21.2.1 Dot product of vectors

Definition 21.13 (Dot Product). If $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$, then the dot product of $\vec{a}$ and $\vec{b}$ is the number $\vec{a} \cdot \vec{b}$ given by

$$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

Theorem 21.14 (Properties of dot product). If $\vec{a}, \vec{b}$ and $\vec{c}$ are vectors in $\mathbb{R}^n$, $n > 0$ and k is a scalar, then

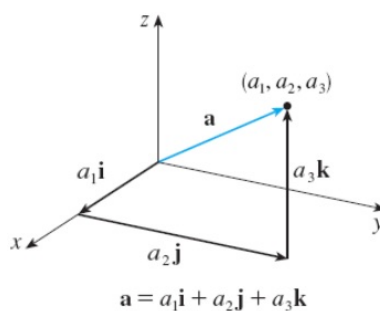


Figure 21.15: Representation of a vector in terms of standard basis vectors.

1. $\vec{a} \cdot \vec{a} = |\vec{a}|^2$
2. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
3. $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
4. $(k\vec{a}) \cdot \vec{b} = k(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (k\vec{b})$
5. $\vec{0} \cdot \vec{a} = \vec{0}$.

Theorem 21.15. If θ is the angle between the vectors $\vec{a}$ and $\vec{b}$, then

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta.$$

Corollary 21.16. Two vectors $\vec{a}$ and $\vec{b}$ are orthogonal if and only if $\vec{a} \cdot \vec{b} = 0$.

Note that, as the standard basis vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ are orthogonal to one another and are unit in length, we have

$$\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1 \quad \text{and} \quad \hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0.$$

Definition 21.17 (Orthogonal projection). The orthogonal projection of v on an arbitrary nonzero vector b can be obtained by

$$\text{proj}_{\vec{b}} \vec{v} = \frac{(\vec{v} \cdot \vec{b})\vec{b}}{|\vec{b}|^2}.$$

Geometrically, if $\vec{b}$ and $\vec{v}$ have a common initial point, then $\text{proj}_{\vec{b}} \vec{v}$ is the vector that is determined when a perpendicular is dropped from the terminal point of $\vec{v}$ to the line through $\vec{b}$; see Figure 21.16.

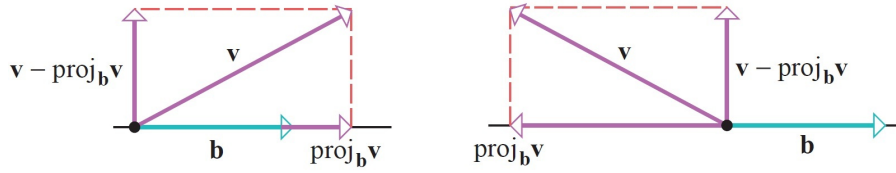


Figure 21.16: Projection of vector $\vec{v}$ on $\vec{b}$. Left: Acute angle between $\vec{v}$ and $\vec{b}$. Right: Obtuse angle between $\vec{v}$ and $\vec{b}$.

21.2.2 Cross product of vectors

Definition 21.18 (Cross product). If $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$, then the cross product of $\vec{a}$ and $\vec{b}$ is the vector $\vec{a} \times \vec{b}$ given by

$$\vec{a} \times \vec{b} = (a_2 b_3 - a_3 b_2, \quad a_3 b_1 - a_1 b_3, \quad a_1 b_2 - a_2 b_1)$$

The cross product $\vec{a} \times \vec{b}$ can be expressed in determinant form as

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

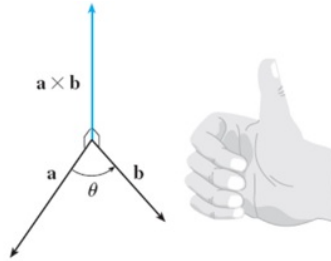


Figure 21.17: right thumb rule for the cross product.

Theorem 21.19. If θ is the angle between the vectors $\vec{a}$ and $\vec{b}$, then

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta.$$

Corollary 21.20. Two vectors $\vec{a}$ and $\vec{b}$ are parallel if and only if $\vec{a} \times \vec{b} = \vec{0}$.

As the standard basis vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ are mutually orthogonal and are unit in length, we have

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0} \quad \text{and} \quad \hat{i} \times \hat{j} = \hat{k}, \quad \hat{j} \times \hat{k} = \hat{i}, \quad \hat{k} \times \hat{i} = \hat{j}.$$

Theorem 21.21 (Properties of cross product). *If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are vectors in $\mathbb{R}^n$, $n > 1$, and k is a scalar, then*

1. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
2. $(k\vec{a}) \times \vec{b} = k(\vec{a} \times \vec{b}) = \vec{a} \times (k\vec{b})$
3. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
4. $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$
5. $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$
6. $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$